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Identification of weakly nonlinearities in multiple coupled oscillators

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Abstract

This paper considers the analysis of free nonlinear response of multi-degree-of-freedom (mdof) vibration systems, demonstrating that system skeleton curves are significantly changed in the presence of coupling. The Hilbert transform vibration decomposition together with the modal-spatial coordinate transform allow precise identification of initial nonlinear characteristics.

1. Introduction

Current research literature presents various techniques for identification and analysis of multi-degree-of-freedom (mdof) nonlinear oscillators. Most researchers recognize that proper signal characterization of the nonlinearities is sufficient for applying system identification. Thus, Ta and Lardies [1] propose the continuous wavelet transform for identifying and quantifying nonlinearities of each vibration mode; the approach in [2] is based on frequency response functions and first-order describing functions, which represent nonlinearities as amplitude-dependent coefficients; Kerschen et al. [3] and Pai and Hu [4] use a popular Empirical Mode Decomposition of vibration signal for nonlinear identification. In the majority of the cases the research states that an a priori spatial nonlinear model with the preliminary solution analysis is required for proper system identification. For example, Raman et al. [5] use an approximate analytical reverse-multi-input/single-output technique for studying how nonlinear equations of motion are governed.

In principle, when the nonlinear detailed model structure and the solution are known, the identification can be turned to a parametric identification problem, where unknown parameters can be derived just by fitting the algebraic expression to data. Those parametric identification techniques, based on the a priori nonlinear model, are only suitable for the chosen model. More preferable are nonparametric identification methods like [6], which do not require an a priori model. Can a nonparametric identification approach be proposed, which enables restoring different nonlinear multiple coupled oscillators? Such an approach should detect and characterize the type and the degree of nonlinearities, unknown vibration model structure and parameters, considering only measured vibration and excitation data.

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The known nonparametric methods developed for nonlinear identification of single-degree-of-freedom (sdof) systems [7,8] are not suitable for mdof systems. What makes the mdof systems so unique? Naturally it is coupling, which forms relations and ties together vibration motions of the linked spatial subsystems. Indeed, if a mdof system is just formed by uncoupled subsystems, described for example by a modal model with independent coordinates, the nonlinear normal modes can be identified separately as a simple composition of sdof systems.

It is well known that coupling moves the modal resonance frequencies away from the initial spatial natural frequencies. So the question arises: does the coupling also change nonlinear skeleton curves of the initial spatial subsystems?

The goal of the current work is to clarify some specific features of the couplings, examine their influence on free vibration of nonlinear multiple coupled oscillators, and explore available engineering options for identification of the initial nonlinear spatial mathematical model of mdof systems. The research focuses on the common phenomena of coupled nonlinear oscillators with typical linear and nonlinear coupling structures.

2. Identification of linear coupled oscillators

Analytical vibration analysis in mechanical engineering deals mainly with second-order equations of motion. These equations combined into a system constitute an initial mathematical spatial model with proper physical parameters (of mass, stiffness and damping properties) describing the dynamic behavior of the test structure. Some typical engineering structures also contain localized nonlinearities (joints, geometric discontinuities, shock absorbers, etc.). A primary objective of the identification process is to derive the initial spatial mathematical model of vibration, including the coupling characteristics and the significant nonlinear elements. For reasons of simplicity we consider a system with only two coupled vibration equations, which completely describe the main properties of mdof systems.

2.1. Spring coupling

For example, the 2-dof linear system includes two linked equations of free motion:

$$\ddot{\varphi} + 2h_{\varphi}\dot{\varphi} + \omega_{\varphi}^2\varphi - \eta_{\varphi}\xi = 0,$$
$$\ddot{\xi} + 2h_{\xi}\dot{\xi} + \omega_{\xi}^2\xi - \eta_{\xi}\varphi = 0, \tag{1}$$

where φ and ξ are the coordinates of coupled vibration motion; $\omega_{\varphi}{}^2$ and $\omega_{\xi}{}^2$ are the spatial (partial) frequency squares, equal to the natural frequency square of the uncoupled (separated, single) equation; h_{φ} and h_{ξ} are the damping coefficients; η_{φ} and η_{ξ} are the coupling spring coefficients. Relations between every spatial frequency and the corresponding vibration amplitude specify the spatial skeleton curve, which in the case of linear oscillators are trivial straight lines.

For further analysis we use substitutions in the form of the analytical signal [7]:

$$X = x + \mathrm{j}\tilde{x}; \quad \dot{X} = X(\dot{A}/A + \mathrm{j}\omega); \quad \ddot{X} = X(\ddot{A}/A - \omega^2 + 2\mathrm{j}\dot{A}\omega/A + \mathrm{j}\dot{\omega}),$$

where $\tilde{x}$ is the Hilbert transform (HT) of x; $A, \dot{A}, \ddot{A}$ are the signal envelope and its derivatives; $\omega, \dot{\omega}$ are the signal instantaneous frequency and its derivative. In reality, the damping coefficients and the derivatives of the envelope are much less than the natural frequencies, so their influence can be ignored ($\dot{A}/A = \ddot{A}/A = 0$). As a result the real part of Eq. (1) will obtain a set of coupled equations

$$\varphi(-\omega^2 + \omega_{\varphi}^2) - \eta_{\varphi}\xi = 0,$$
$$\xi(-\omega^2 + \omega_{\xi}^2) - \eta_{\xi}\varphi = 0 \tag{2}$$

whose determinant is equal to zero $\omega^4 - \omega^2(\omega_{\varphi}^2 + \omega_{\xi}^2) + \omega_{\varphi}^2\omega_{\xi}^2 - \eta_{\varphi}\eta_{\xi} = 0$ produces a known biquadratic equation for calculation of the modal normal frequencies:

$$\omega_{1,2}^2 = 1/2\{\omega_{\varphi}^2 + \omega_{\xi}^2 \pm [(\omega_{\varphi}^2 - \omega_{\xi}^2)^2 + 4\eta_{\varphi}\eta_{\xi}]^{1/2}\}. \tag{3}$$

The relations between the modal frequencies versus the initial spatial frequencies (3), known as Wien's graph, illustrate the fact that the initial spatial frequencies always lie between the modal frequencies. So each natural frequency differs from the initial spatial (partial) subsystem natural frequency. In other words, the obtained modal frequencies are not those of the individual component systems. The difference between the spatial and modal frequencies, controlled by coupling coefficients, is governed by the following decoupling coordinate transformation.

Each equation of the system Eq. (2) characterizes the coefficient of amplitude ratio (distribution) ψ between oscillations of different coordinates at every modal frequency:

$$(\xi/\varphi)_1 = (\omega_\varphi^2 - \omega_1^2)/\eta_\varphi = \psi_1,$$
$$(\varphi/\xi)_2 = (\omega_\varphi^2 - \omega_2^2)/\eta_\varphi = \psi_2. \quad (4)$$

These relative amplitudes corresponding to every modal frequency, known as the mode shapes, are also the fundamental inherent properties of a freely vibrating, undamped mdof system. The process of calculating modal parameters from the initial spatial system refers to analytical modal analysis, which transforms or decouples the spatial system into a system of several equations, one for each single mode of vibration. This means that the calculated modes of vibration effectively uncouple the dynamic equations of motion according to the following relationships between the initial coordinates φ, ξ and the normal coordinates x, y [9]:

$$\varphi = x + y, \quad \xi = \psi_1 x + \psi_2 y; \quad x = (\xi - \varphi\psi_2)/(\psi_1 - \psi_2), \quad y = (\psi_1\varphi - \xi)/(\psi_1 - \psi_2). \quad (5)$$

The initial coordinate of every dof is often a real physical coordinate (direction) of vibration measurement, whereas the normal coordinate is a virtual (abstract) coordinate. The real oscillations of the masses can be written as linear combinations of the normal modes.

Expressions Eq. (5) show that coupling as the important common property completely defines relations between the modal and spatial parameters. What's more, the coupling also defines the energy exchange and the time of the energy transfer between the participating subsystems. In coupled subsystems, as the coupling strength is increased from zero, the oscillations affect each other more and more. Thus, the motion generated from one coordinate will appear stronger in the vibration of other coupled coordinates. So the coupling gives rise to the multiplicity of natural oscillations in observed vibration motion. The coupling strength coefficient σ characterises the degree of coupling between two subsystems [9]: $\sigma = \left(2\sqrt{\eta_\varphi\eta_\xi}\Big/|\omega_\varphi^2 - \omega_\xi^2|\right)$. Note that damping coupling, no more than the small partial damping, has practically no influence on the coupled vibration, natural frequencies and modal shapes.

2.2. Reconstruction of coupling coefficients

Traditionally the model in question is only a modal model because the modal properties of the system most closely describe the dynamic behavior observed in the tests. For purposes of identification of mdof systems, this is not sufficient. A modal test based entirely on measured vibration data should lead to further inverse reconstruction of a spatial coupled mathematical model based on mass, stiffness, damping properties, and coupling stiffness forces, which have a physical meaning. During the vibration test the modal modes are excited, so the modal frequencies ω_i together with the mode shapes ψ_i can be observed and estimated. It is enough to reconstruct the initial partial natural frequencies and the spring couplings of the spatial model. In effect, two equations from Eq. (3) and two equations from Eq. (4) together involve only four unknowns and the general case of n-dof yields $2n$ linear equations with $2n$ unknowns. So the direct solution of the linear system Eqs. (3) and (4) returns the initial parameters of the spatial model:

$$\omega_\varphi^2 = (\psi_1\omega_2^2 - \psi_2\omega_1^2)/(\psi_1 - \psi_2); \quad \omega_\xi^2 = (\psi_1\omega_1^2 - \psi_2\omega_2^2)/(\psi_1 - \psi_2),$$
$$\eta_\varphi = (\omega_2^2 - \omega_1^2)/(\psi_1 - \psi_2); \quad \eta_\xi = \psi_1\psi_2(\omega_1^2 - \omega_2^2)/(\psi_1 - \psi_2), \quad (6)$$

where ω_φ and ω_ξ are the resultant spatial frequency squares, ω_φ and ω_ξ are the measured mode shapes (amplitude ratios), and ω_1^2 and ω_2^2 are the measured modal frequency squares. The above formulas allow deriving back an initial mathematical spatial model to describe the dynamic behavior of the test system without an a priori model description.

3. Identification of weakly nonlinear coupled vibration oscillators

Typically, every nonlinear equation expressing vibration motion has a fixed structure. This structure classically includes three independent elements: restoring elastics force (stiffness, spring) as a nonlinear function of displacement (position), damping force (friction) as a nonlinear function of velocity (the first derivative of position with respect to time), and inertial force proportional to acceleration (the second derivative of position with respect to time). Every independent restoring and damping force element is an a priori unknown nonlinear function of motion, as for example hardening or softening spring, dry or turbulent friction.

Nonparametric identification of nonlinear vibration oscillators as a typical dynamics inverse problem deals with a priori unknown nonlinear restoring and damping functions. The investigated vibration system with unknown restoring and damping forces moves under (or without) excitation force. By observation (experiment), we acquire knowledge of the position and/or velocity of the object as well as the excitation at several known instants of time. The nonparametric identification will determine the initial nonlinear restoring and damping forces. In the case of free vibration we have only the output signal—the vibration of the oscillators; in the case of forced vibration we deal also with the input excitation.

Nonlinear vibration mdof systems can consist of essential nonlinear oscillators joined with linear couplings or of linear oscillators coupled with essential nonlinear attachments. They could also represent a united case of both the nonlinear oscillators and the nonlinear couplings acting in combination.

3.1. Identification of linear coupled nonlinear oscillators

In general, nonlinear systems composed of several masses; nonlinear springs and dampers require more complicated representation. First, let us consider equations of motion for a coupled 2-dof system, wherein the stiffness nonlinearity depends only on the displacement:

$$\begin{aligned} &\ddot{\varphi} + \omega_\varphi^2 \varphi + \alpha\varphi^3 - \eta_\varphi \xi = 0, \\ &\ddot{\xi} + \omega_\xi^2 \xi - \eta_\xi \varphi = 0, \end{aligned} \tag{7}$$

here the first equation includes a simple, relatively weak nonlinear cubic stiffness $\alpha\varphi^3$ corresponding to the initial linearized spatial skeleton curve $\omega_0^2(A) = {\omega_\varphi}^2 + \frac{3}{4}\alpha A^2$ [7]. This model combines weakly nonlinear oscillators (without bifurcations, jumps and chaotic behavior) that have a slow varying solution in the time domain.

Again for further analysis, we will use substitutions in the form of the analytical signal considering the overlapping spectra property of the HT of nonlinear functions. The HT substitution establishes direct relationships between the parameters of the initial differential equations and the instantaneous amplitude and frequency of the vibration response. The HT reduction allows direct construction of an approximate solution defined as a single quasi-harmonic with slow varying amplitude and frequency. In essence, the HT approach is just an alternative to well-known linearization methods such as the harmonic balance linearization. The HT of the harmonics cube will contains the first and third components. Hence, neglecting high-tripled frequency yields [7] $H[A\cos^3\theta] = A^3(3\sin\theta + \sin 3\theta)/4 \approx \frac{3}{4}A^3 \sin\theta$.

The real part of Eq. (7) takes the following form:

$$\begin{aligned} &\varphi(-\omega^2 + \omega_\varphi^2 + \tfrac{3}{4}\alpha A_\varphi^2) - \eta_\varphi \xi = 0, \\ &\xi(-\omega^2 + \omega_\xi^2) - \eta_\xi \varphi = 0. \end{aligned}$$

We will not solve the obtained nonlinear system, but only analyze the corresponding biquadratic equation for the fixed amplitude $\omega^4 - \omega^2(\omega_\varphi^2 + \omega_\xi^2 + \frac{3}{4}\alpha A_\varphi^2) + \omega_\varphi^2\omega_\xi^2 + \frac{3}{4}\alpha A_\varphi^2\omega_\xi^2 - \eta_\varphi\eta_\xi = 0$.

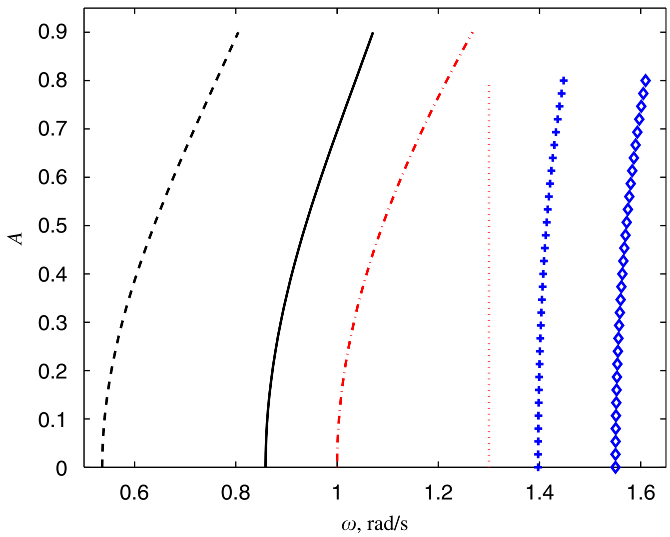

Fig. 1. Influence of the coupling coefficient ($\eta = \eta_\varphi = \eta_\xi$) on the skeleton curves of the nonlinear 2-dof vibration system. The first mode ω_1 ($\omega_\varphi = 1$): $\eta = 1$ (- -), $\eta = 0.5$ (—), $\eta = 0$ (-·-); the second mode ω_2 ($\omega_\xi = 1.3$): $\eta = 0$ (···), $\eta = 0.5$ (+), $\eta = 1$ (◇).

It results in the modal frequencies $\omega_{1,\,2}(A)$ of normalized oscillations as functions of the amplitude:

$$\omega^2_{1,2}(A) = \tfrac{1}{2}\{\omega^2_\varphi + \omega^2_\xi + \tfrac{3}{4}\alpha A^2_\varphi \pm [(\omega^2_\varphi - \omega^2_\xi)^2 + 4\eta_\varphi\eta_\xi + \tfrac{9}{16}\alpha^2 A^4_\varphi + \tfrac{3}{2}\alpha A^2_\varphi(\omega^2_\varphi - \omega^2_\xi)]^{1/2}\}. \tag{8}$$

These varying modal natural frequencies of the coupled subsystems are also fundamental properties, named modal skeleton curves, which are independent of the choice of coordinates or external excitation. But again the modal and the corresponding spatial skeleton curves differ from each other.

The modal skeleton curves obtained from Eq. (8) are shown in Fig. 1. They capture the effect of nonlinear behavior, when coupling again pushes apart the spatial skeleton frequencies, and the low frequency becomes lower and the high frequency becomes higher. But now the obtained normal frequencies, besides their dependencies on the coupling, are also functions of the vibration amplitude, so every normal frequency will form the corresponding normal skeleton curve. Analysis of Eq. (8) shows that the coupling transfers a single spatial nonlinearity all over coupled modal skeleton curves controlling the strength of the nonlinear behavior. As the coupling coefficient is increased from zero, all other coupled skeleton curves will be rearranged from the trivial vertical lines to a more and more nonlinear form. This new result from Eq. (8) means that the coupling smears out (spread) a nonlinear effect all over all modal coordinates.

Further analysis of Eq. (8) shows that the initial nonlinearity concurrently influences the entire normal skeleton curves with the same tendency. For example, an initial hardening stiffness will also appear as a hardening in every normal skeleton curves, and a softening will exhibit a softening. But every normal skeleton will only be qualitatively similar in appearance to the initial spatial skeleton curve. However, the normal and the initial spatial skeleton curves will differ quantitatively from each other. To identify the initial spatial skeleton curve we need to consider the initial spatial frequencies for the nonlinear case.

The coefficients of amplitude distribution in a nonlinear system will also vary as functions of amplitude: $(\xi/\varphi)_1 = (\omega^2_\varphi - \omega^2_1 + \tfrac{3}{4}\alpha A^2_\varphi)/\eta_\varphi = \psi_1$; $(\xi/\varphi)_2 = (\omega^2_\varphi - \omega^2_2 + \tfrac{3}{4}\alpha A^2_\varphi)/\eta_\varphi = \psi_2$. Such an effect of the amplitude variation of the mode shapes indicates the presence of a nonlinear element, the initial spatial equation. From the coefficients of amplitude distribution above and the modal skeleton curves (8), we can restore both the coupling coefficients $\eta_{\varphi,\xi}$ and the initial spatial skeleton curves $\omega^2_{\varphi,\xi}(A)$:

$$x\omega^2_\varphi(A) = (\psi_1\omega^2_2 - \psi_2\omega^2_1)/(\psi_1 - \psi_2) + \tfrac{3}{4}\alpha A^2; \quad \omega^2_\xi(A) = (\psi_1\omega^2_1 - \psi_2\omega^2_2)/(\psi_1 - \psi_2),$$
$$\eta_\varphi = (\omega^2_2 - \omega^2_1)/(\psi_1 - \psi_2); \quad \eta_\xi = \psi_1\psi_2(\omega^2_1 - \omega^2_2)/(\psi_1 - \psi_2).$$

The obtained spatial skeleton curve from the first equation $\omega_\varphi^2(A)$ indeed involves initial localized nonlinearity with exact physical characteristics of the nonlinear element $\frac{3}{4}\alpha A^2$. As this takes place, the spatial skeleton curve of the second linear equation $\omega_\xi{}^2(A)$ remains linear. This means that the restored spatial skeleton curves will retain the same nonlinear behavior when the initial local model involves a nonlinear element and the restored spatial skeleton curves will stay linear if the initial local model does not include nonlinearities.

3.2. Identification of coupled linear oscillators with nonlinear couplings

Consider another mdof nonlinear system, now governed by two linear oscillators coupled with a nonlinear element:

$$\begin{aligned} &\ddot{\varphi} + \omega_\varphi^2\varphi - \eta_\varphi\xi - \beta\xi^3 = 0, \\ &\ddot{\xi} + \omega_\xi^2\xi - \eta_\xi\varphi = 0, \end{aligned} \tag{9}$$

here the mathematical model can be expressed as where the coefficient η_φ represents the linear spring coupling, while the coefficient β accounts for the nonlinear cubic stiffness coupling due to the spatial coordinate ξ.

The first harmonics of the HT of cube function takes the form [7]: $H[A\cos^3\theta] \approx \frac{3}{4}A^3\sin\theta$, so the real part of Eq. (9) can be written as a set of equations:

$$\varphi(-\omega^2 + \omega_\varphi^2) - \xi(\eta_\varphi + \tfrac{3}{4}\beta A^2) = 0; \quad \xi(-\omega^2 + \omega_\xi^2) - \eta_\xi\varphi = 0. \tag{10}$$

The determinant of the above equation equal to zero gives the following modal normal frequencies as functions of amplitude:

$$\omega_{1,2}^2(A) = \tfrac{1}{2}\{\omega_\varphi^2 + \omega_\xi^2 \pm [(\omega_\varphi^2 - \omega_\xi^2)^2 + 4\eta_\varphi\eta_\xi + 3\eta_\xi\beta A^2]^{1/2}\}. \tag{11}$$

The obtained modal skeleton curves depend on the amplitude and are nonlinear functions in spite of the linear nature of the oscillators under consideration. The mode shapes derived from Eq. (10) are also nonlinear functions:

$$(\xi/\varphi)_1 = (\omega_\varphi^2 - \omega_1^2)/(\eta_\varphi + \tfrac{3}{4}\beta A_\varphi^2) = \psi_1; \quad (\xi/\varphi)_2 = (\omega_\varphi^2 - \omega_2^2)/(\eta_\varphi + \tfrac{3}{4}\beta A_\varphi^2) = \psi_2. \tag{12}$$

Combining four equations from Eqs. (11) and (12) we get back the spatial natural frequencies and coupling coefficients:

$$\begin{aligned} &\omega_\varphi^2(A) = (\psi_1\omega_2^2 - \psi_2\omega_1^2)/(\psi_1 - \psi_2); \quad \omega_\xi^2(A) = (\psi_1\omega_1^2 - \psi_2\omega_2^2)/(\psi_1 - \psi_2), \\ &\eta_\varphi(A) = (\omega_2^2 - \omega_1^2)/(\psi_1 - \psi_2) - \tfrac{3}{4}\beta A^2; \quad \eta_\xi = \psi_1\psi_2(\omega_1^2 - \omega_2^2)/(\psi_1 - \psi_2). \end{aligned}$$

Note that the restored coupling $\eta_\varphi(A)$ correctly represents the initial nonlinear coupling function.

In the general case, mdof vibration systems can have both types of nonlinearities together—the oscillator nonlinear stiffness itself and the nonlinear stiffness coupling, acting simultaneously. In such a case of combined nonlinear spring and nonlinear coupling, the modal skeleton curves will have a rather complicated form. Nevertheless, as was shown, the restored spatial skeleton curves and the coupling coefficients will return the correct initial spatial characteristics of the system.

4. HT nonstationary vibration decomposition and analysis

The considered estimation of the spatial model characteristics requires two groups of amplitude varying data: modal frequencies and modal shapes. New nonstationary signal decomposition methods based on the HT [11,12] work as a kind of adaptive filters in the time–frequency domain and make it possible to obtain the required amplitude varying data to use for further identification.

In general, the vibration of a coordinate of nonlinear multiple coupled oscillators simultaneously demonstrates two different physical natures of vibration motions. The first multicomponent oscillation is just the presence of partial motion from the coupled subsystems. These coupled components also exist in linear mdof systems. The second group of multicomponent oscillation is associated with intricate nonlinear relationships in restoring and damping functions, which produce nonlinear harmonic and intermodulation distortions. Actually, a real motion of the nonlinear systems contains several main or principal quasi-harmonic solutions together with an infinite number of multiple high-frequency superharmonics [8]. In this paper, we consider only the first group of motion that includes the main primary multicomponent system solution.

4.1. HT signal decomposition

In the case of multiple coupled vibration oscillators, the HT signal decomposition takes apart every normal mode. It is able to get rid of the mode mixing phenomenon and attempting to purify and clean every vibration mode signal. The HT decompositions do not require preliminary band-pass filtration of the signal in order to pick out each mode of interest and to reject all the others. After using the decoupling technique, we will have several corresponding decoupled vibration motions.

The idea of the HT methods is to decompose an initial wideband oscillation $x(t)$ into a sum of elementary components with slow varying instantaneous amplitude and frequency, so that $x(t) = \sum_l a_l(t)\cos(\int \omega_l(t)\mathrm{d}t)$, where $a_1(t)$ is the instantaneous amplitudes and $\omega_l(t)$ is the instantaneous frequencies of the l component. Thus, the obtained instantaneous frequency of each component will correspond to decoupled modal natural frequency, and the instantaneous amplitude ratio will match the normal mode shapes—all as functions of time t.

4.2. Modal skeleton curve estimation

Every decomposed modal vibration is a solution of the corresponding modal sdof second-order system having the nonlinear elastic (restoring) force characteristics $k(x)$. The nonlinear restoring force can be represented as the multiplication of a varying nonlinear natural frequency $\omega_0^2(x)$ and the nonlinear oscillator solution x

$$\ddot{x} + h_0(\dot{x}) + k(x) = \ddot{x} + h_0(\dot{x}) + \omega_0^2(x)x = 0.$$

The instantaneous undamped modal natural frequency and the instantaneous damping coefficient of the tested oscillator are estimated according to the FREEVIB formulas [7]:

$$\omega_0^2(t) = \omega^2 - \frac{\ddot{A}}{A} + \frac{2\dot{A}^2}{A^2} + \frac{\dot{A}\dot{\omega}}{A\omega},$$
$$h_0(t) = -\frac{\dot{A}}{A} - \frac{\dot{\omega}}{2\omega},$$

where $A(t)$ and $\omega(t)$ are the envelope and the instantaneous frequency of the modal vibration.

4.3. Mode shape estimation

In nonlinear oscillators, the envelope of every decomposed nonstationary modal component varies in time, so the varying modal shape of the i mode takes the form $\psi_i(A) = A_{\xi i}/A_{\varphi i}\cos\theta_i$. Here $A_{\xi i}$ is the envelope of the i decomposed modal component of the spatial coordinate ξ, $A_{\varphi i}$ is the envelope of the same frequency modal component of the next spatial coordinate φ, and θ_i is the phase between the modal vibration components ξ_i and φ_i.

The HT decomposition allows detecting and isolating the modal frequencies and the modal shapes even in the case of time varying amplitude (mode), changing frequency, envelope decay, and phase variation in time for each isolated mode. Thus, the recent achievements in nonstationary signal decomposition [10,11] open a way for new combined analysis and identification of both the linear and nonlinear mdof vibration systems. Next, we describe the common identification scheme and consider some examples of identification of weakly nonlinear multiple vibration oscillators.

5. Description of the identification scheme

The main idea of the proposed identification is to apply the known linear inverse transformation between the modal to the spatial coordinates to obtain the correct initial spatial nonlinear characteristics. The free vibration of an undamped structure which is assumed to be linear and proximately discretized for n-dof can be described by the spatial equations of motion: $[\mathbf{M}]\{\ddot{\boldsymbol{\gamma}}\}+[\mathbf{K}]\{\boldsymbol{\gamma}\}=\mathbf{0}$, where $[\mathbf{M}]$, $[\mathbf{K}]$ and $\{\boldsymbol{\gamma}\}$ are matrices of the spatial mass, stiffness and the vector of the displacement. If the number of measured modes is equal to the number of measured coordinates n, the transformation can be written as [12]:

$$[\mathbf{M}]=[\mathbf{\Phi}]^{-\mathrm{T}},\ [\mathbf{K}]=[\mathbf{\Phi}]^{-\mathrm{T}}[\boldsymbol{\lambda}_r^2][\mathbf{\Phi}]^{-1}, \tag{13}$$

where $[\mathbf{\Phi}]=[\boldsymbol{\psi}][\mathbf{m}_r]^{-1/2}$ is the mass-normalised eigenvectors, $\boldsymbol{\psi}$ is the mode shape, and $\mathbf{m}_r$ is the modal mass.

In addition to reconstruction of the linear mdof spatial model, the coordinate transformation enables reconstruction of the initial nonlinear elements. The actual number of nonlinear elements can be small, but their overall nonlinear behavior may be significant. Provided modal analysis bases on the HT signal decomposition will make available estimation of the modal skeleton curves and the mode shapes of every nonlinear mode of the coupled vibration oscillator. But these nonlinear modal characteristics are shifted because of the coupling, and moreover, they differ quantitatively from the initial identified characteristics. Only the use of coordinate transformations, such as algebraic formulas Eq. (13), allows reconstruction of the initial nonlinear characteristics. For different nonlinear vibration systems, the mentioned coupling reconstruction is a nonparametric identification technique that does not require a priori model description.

The spring force characteristics $k(x)$ intended for identification are defined as the multiplication of two phasors: $K=\omega_0^2X$, where ω_0^2 is the varying nonlinear natural frequency, and X is the displacement of the

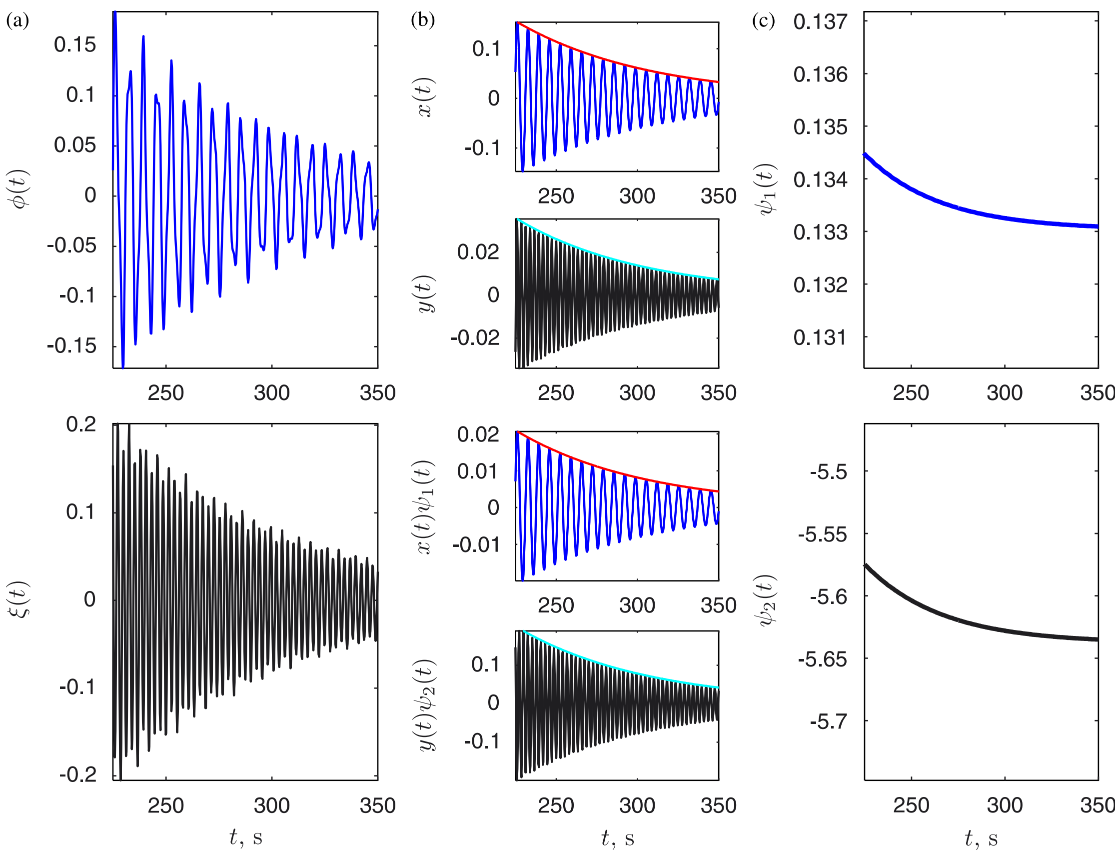

Fig. 2. Free vibration of two coupled Duffing oscillators: the first $\varphi(t)$ and the second $\xi(t)$ spatial coordinate vibration (a), the HT decomposed normal coordinates (b), mode shapes of two modes (c).

vibration in the signal analytic form. For a complex product, the magnitudes are multiplied and the angles are added together, so the following "envelope" expression returns the initial spatial static force characteristics:

$$k(x) = \begin{cases} \omega_0^2(A)A, & x>0, \\ -\omega_0^2(A)A, & x<0, \end{cases}$$

$$h(\dot{x})\dot{x} = \begin{cases} h_0(a_{\dot{x}})a_{\dot{x}}, & \dot{x}>0, \\ -h_0(a_{\dot{x}})a_{\dot{x}}, & \dot{x}<0, \end{cases}$$

where $a_{\dot{x}}$ is the envelope of the velocity. The estimated average natural frequency and the average damping function include the main information about the initial nonlinear elastic and damping characteristics.

The proposed identification is a three-stage method, that includes the following procedures: (a) multipoint vibration measurements of every mass belonging to mdof system; (b) HT signal decomposition of the measured vibrations into modal (normal) vibration components and estimation the corresponding modal skeleton curves and mode shapes; (c) estimation of the initial spatial skeleton curves and couplings and reconstruction of the spatial nonlinear differential equations of the initial model. The proposed nonparametric identification method is dedicated primarily to stable weakly nonlinear oscillators with quasi- and almost periodic oscillating-like solutions.

6. Simulation examples

6.1. Model 1. Nonlinear oscillators with linear coupling

As a first example, consider a system of two coupled nonlinear equations, where the first (driving) oscillator includes the hardening and the second (driven) oscillator includes a softening nonlinear spring element:

$$\ddot{\varphi} + 0.05\dot{\varphi} + \varphi + \varphi^3 - 0.8\xi = 0, \varphi_0 = 4.0,$$
$$\ddot{\xi} + 0.05\dot{\xi} + 5.4\xi - 0.5\xi^3 - 0.6\varphi = 0.$$

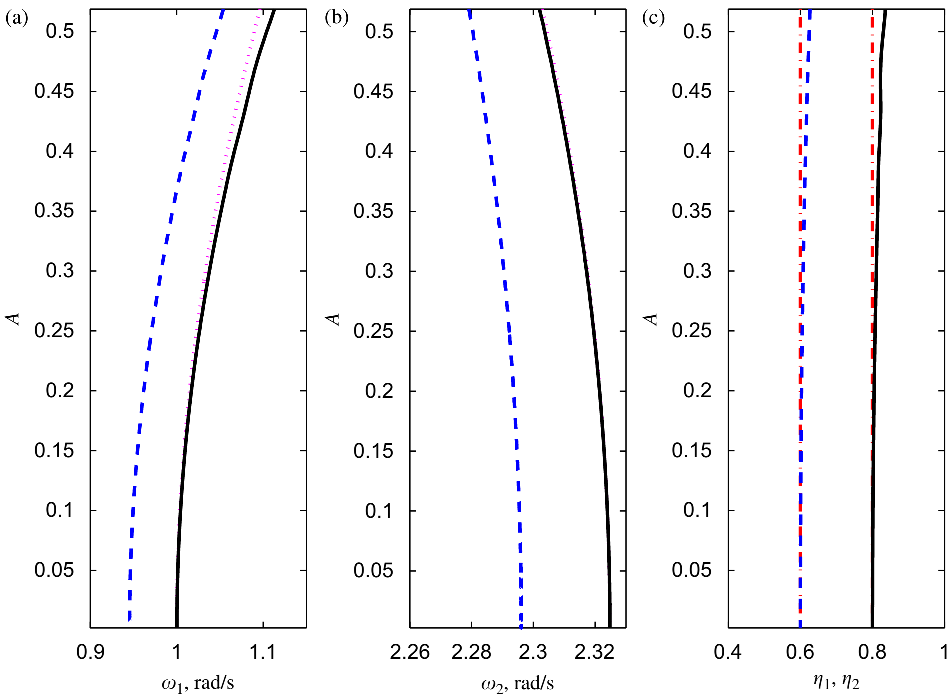

Fig. 3. Skeleton curves of the first hardening stiffness mode (a): the modal (- -), the identified spatial (–), the initial spatial (···); skeleton curves of the second softening mode (b): the modal (- -), the identified spatial (–), the initial spatial (···); the coupling of nonlinear modes (c): the first equation identified spatial (- -), the second equation identified spatial (–), the initials (- · -).

The above system of equations was numerically solved using the fourth-order Runge–Kutta method with a time step of $\Delta t = 0.25$ s. The signal-modulated waveforms from every spatial coordinate are shown in Fig. 2(a), and the decomposed normal smooth components according to the HT method [11] are shown in Fig. 2(b). The estimated ratios between the corresponding decomposed normal envelopes as the modes shape of every nonlinear mode (Fig. 2(b)) are plotted in Fig. 2(c). The estimated mode shapes depend on the amplitude, which at once exhibits the nonlinear behavior of the vibration oscillator. In effect, the estimated modal skeleton curves (Fig. 3(a, b), dashed line) are nonlinear, but both the first hardening and the second softening are far away from the initial spatial skeleton curves (Fig. 3(a, b), dotted line). Only the restored spatial skeleton curves (Fig. 3(a, b), bold line) are very close to the initial skeleton curves. In fact, least-squares fitting data of the resultant identified spatial skeleton curves (Fig. 3(a, b), bold line) returns the nonlinear stiffness coefficient α equal to 0.844 (the initial α for the first mode is equal to 0.75) and to -0.39 (the initial α for the second mode is equal to -0.375), that differs by less than 12% and 4% from the initial nonlinear stiffness coefficient, correspondingly. Both restored coupling static characteristics (Fig. 3(c), bold lines) are close to the initial straight vertical lines (Fig. 3(c), dot-dash lines).

6.2. Model 2. Oscillators with nonlinear coupling

An example of a vibration oscillator considered here is a 2-dof system with linear equations of motion and two opposite cubic nonlinear couplings, respectively:

$$\ddot{\varphi} + 0.01\dot{\varphi} + \varphi - 0.4\xi - \xi^3 = 0, \quad \varphi_0 = 0.5,$$
$$\ddot{\xi} + 0.01\dot{\xi} + 2.88\xi - 0.3\varphi - 0.5\varphi^3 = 0, \quad \xi_0 = 0.7.$$

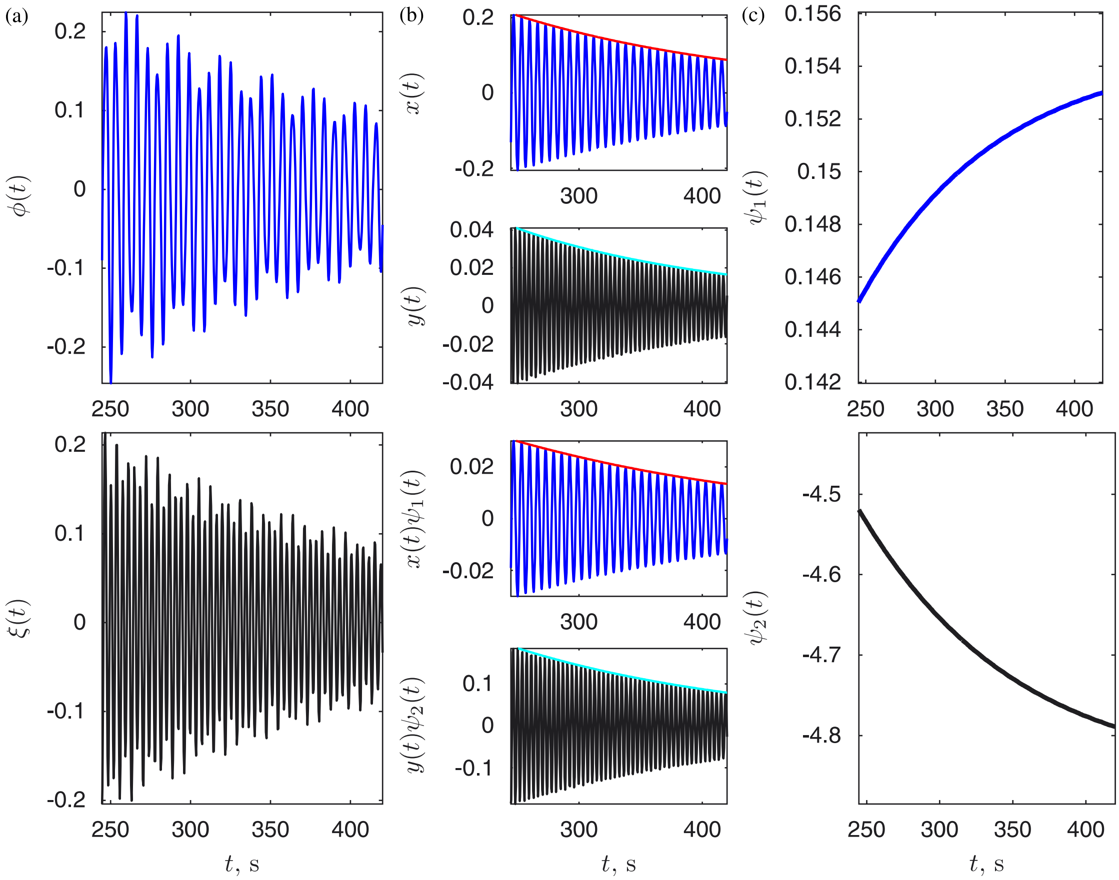

Fig. 4. Free vibration of two oscillators with nonlinear couplings: the first $\varphi(t)$ and the second $\xi(t)$ spatial coordinate vibration (a), the HT decomposed normal coordinates (b), mode shapes of two modes (c).

Simulated free vibration is shown in Fig. 4(a). The decoupled modal component after the HT signal decomposition is illustrated in Fig. 4(b). The corresponding mode shapes (Fig. 4(c)) indicates a nonlinear type of motion. Both obtained spatial skeleton curves of linear oscillators (Fig. 5(a, b), bold lines) practically coincide with the initial vertical straight lines. Every restored coupling static characteristic repeats the corresponding initial cubic spring coupling subsystem (Fig. 5(c)). Thus, the least-squares fitting data of the identified nonlinear first mode coupling characteristics to a polynomial model returns the nonlinear coefficient β equal to 0.79 when the initial β is equal to 0.75. The estimated nonlinear second mode-coupling coefficient β is equal to -0.42 when the initial β for the second mode is equal to -0.375, which differs about 5–10% from the initial nonlinear coefficient values. The comparison between the identified and the initial characteristics shows that the proposed approach enables precise estimation of actual oscillator nonlinearities.

6.3. Model 3. Self-excited oscillators with coupling

The next example combines two van der Pol equations with linear coupling:

$$\ddot{\varphi} + 0.2\dot{\varphi}(\varphi^2 - 1) + \varphi - 0.8\xi = 0, \qquad \varphi_0 = 0.01,$$
$$\ddot{\xi} + 0.3\dot{\xi}(\xi^2 - 1) + 1.88\xi - 0.8\varphi = 0, \quad \xi_0 = 0.02.$$

Every van der Pol equation contains a bilinear cross-term with multiplication of two variables: the displacement squared and the velocity. Due to nonzero initial conditions and due to the presence of unstable damping terms, both coordinates of the tested model immediately display increasing self-excited periodic motion (Fig. 6(a)). After transient increasing motion, the observed steady-state solutions include a combination of self-excited vibrations generated by every coordinate.

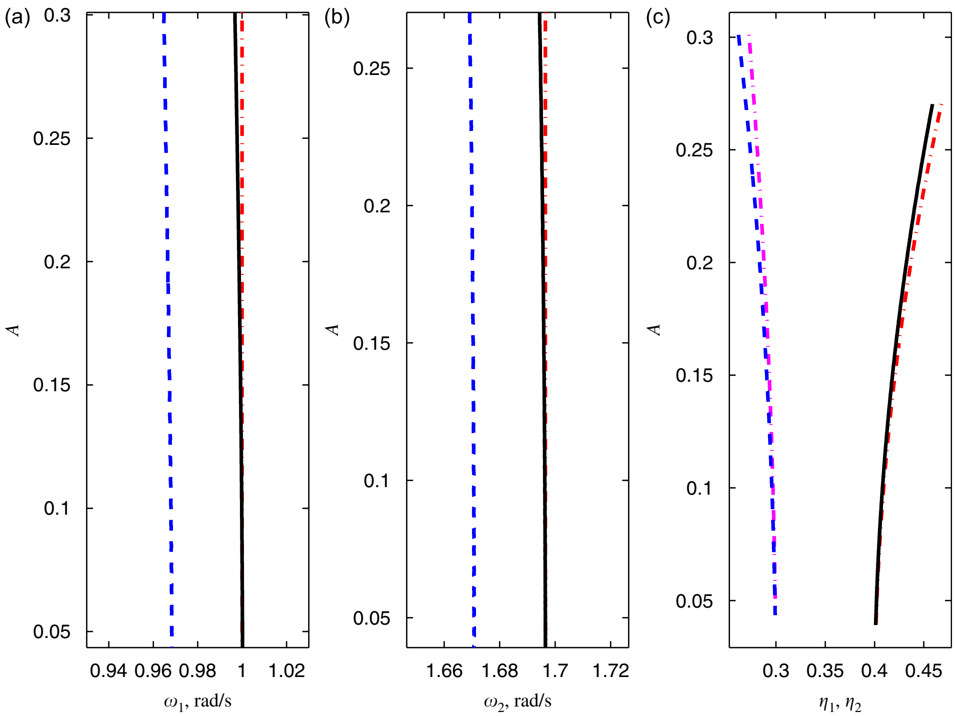

Fig. 5. Skeleton curves of the first mode with nonlinear couplings (a): the modal (- -), the identified spatial (–), the initial spatial (···); skeleton curves of the second mode with nonlinear couplings (b): the modal (- -), the identified spatial (–), the initial spatial (···); the coupling of nonlinear modes (c): the first equation identified spatial (- -), the second equation identified spatial (–), the initials (- · -).

Application of the HT signal decomposition shows these two separated components with different frequencies (Fig. 6(b)). The first component $x(t)$ is generated by the first van der Pol equation, and the second component $y(t)$ by the second equation. The amplitude level of both components depends mainly on the initial bounding friction coefficient of the coordinate squared and not on the coupling coefficients. This means that the estimated ratios between the decomposed normal envelopes (Fig. 6(c)) do not describe the regular mode shapes; nevertheless, the existing coupling has an impact on the modal skeleton curves, shifting them from the initial spatial frequencies (Fig. 7(a, b)).

Appling the HT identification method FREEVIB to the transient increasing motion of every coordinate will return the initial nonlinear friction force characteristics in the form $\dot{x}(x^2 - 1)$ typical of the van der Pol equation (Fig. 7(c, d)).

7. Conclusions

A single nonlinear element presented only in a single specific equation of mdof vibration system will be noticed in the vibration of all coupled coordinates. The resulting behavior is expected due to the dynamic interaction between coupled vibration subsystems. The observed modal nonlinearity does not correspond to the initial nonlinearity of the spatial model. The observed modal skeleton curve will be only similar in appearance to the initial spatial skeleton curve. However, the modal and the initial spatial skeleton curves differ quantitatively from each other. To identify the initial spatial skeleton curve we suggest applying the

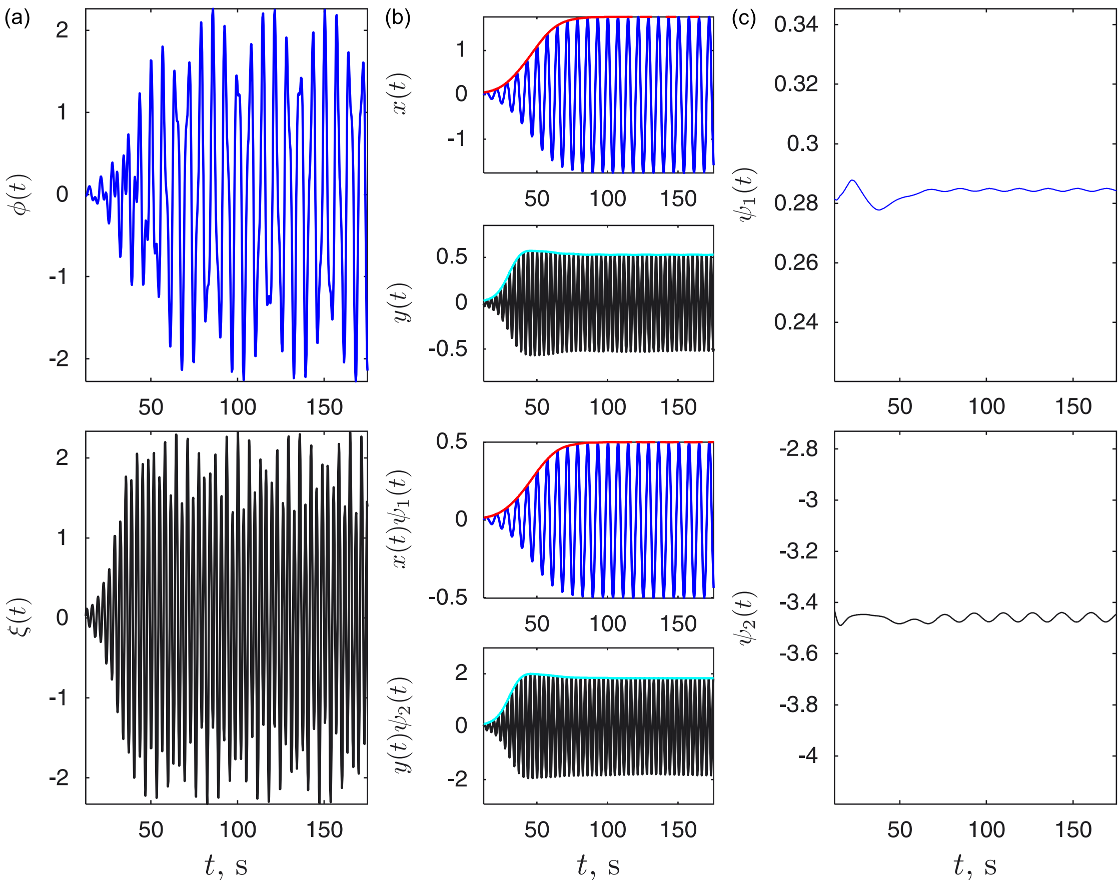

Fig. 6. Self-excited vibration of two coupled van der Pol oscillators: the first $\varphi(t)$ and the second $\xi(t)$ spatial coordinate vibration (a), the HT decomposed normal coordinates (b), mode shapes of two modes (c).

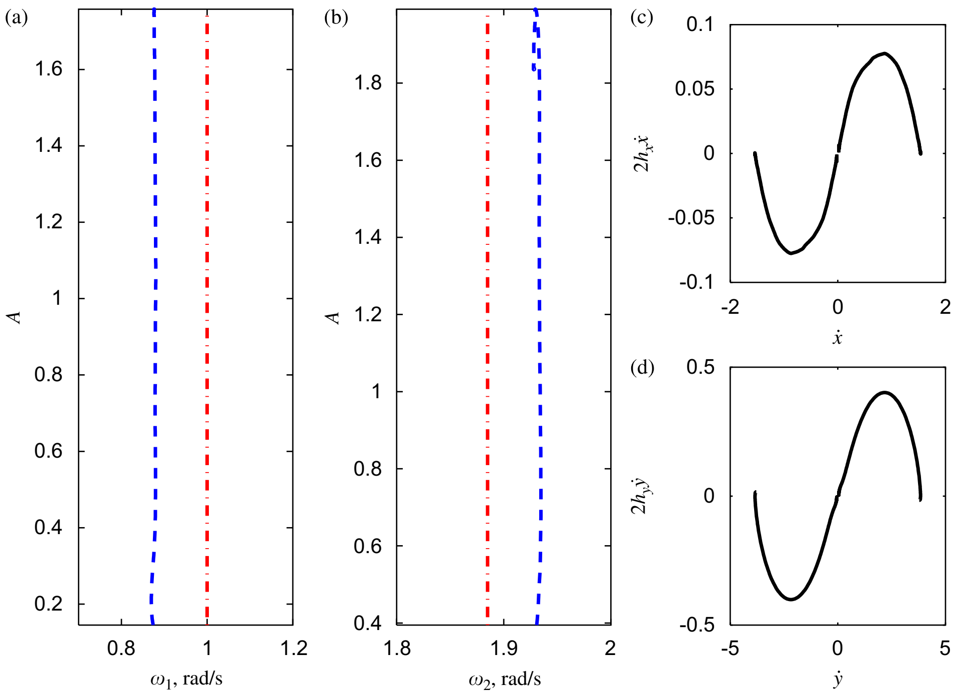

Fig. 7. Skeleton curves of the first mode (a): the modal (- -),the initial spatial (·—·); skeleton curves of the second mode (b): the modal (- -), the initial spatial (·—·); the identified nonlinear friction force of the first van der Pol oscillator (c); the identified nonlinear friction force of the second van der Pol oscillator (d).

modal-spatial coordinate transformation together with the Hilbert transform vibration decomposition. The estimated nonlinearities can be actually quantified and included in the dynamics model. The proposed identification method is a nonparametric method, which does not require a priori consideration of the nonlinear nature of the vibration oscillator. The method is recommended for nonlinear parameter identification, including determination of the system skeleton curve (backbone) and the static coupling force characteristics.

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